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More congruences for central binomial coefficients

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ABSTRACT

We present several congruences for sums of the type $\sum_{k=1}^{p-1} m^k k^{-r} \binom{2k}{k}^{-1}$, modulo a power of a prime p. They bear interesting similarities with known evaluations for the corresponding infinite series.

1. Introduction

In 1979, Apèry [2] proved that $\zeta(3)$ was irrational starting from the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} = -\frac{2}{5}\zeta(3).$$

In the same paper he also pointed out that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \binom{2k}{k}^{-1} = \frac{1}{3}\zeta(2),$$

which has been known since the nineteenth century. Here we present finite analogues of the above identities, in the form of congruences modulo a prime. For any prime $p > 5$ we prove that

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$$\sum_{k=1}^{p-1} \frac{1}{k^2}\binom{2k}{k}^{-1} \equiv \frac{1}{3}\,\frac{H(1)}{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3}\binom{2k}{k}^{-1} \equiv -\frac{2}{5}\,\frac{H(1)}{p^2} \pmod{p^3},$$

where $H(1)=\sum_{k=1}^{p-1} 1/k$. Note that $H(1)\equiv 0 \pmod{p^2}$ according to Wolstenholme's theorem.

After some preliminary results, in the last section we present proofs of the above congruences, as well as proofs of the *dual* congruences

$$\sum_{k=1}^{p-1} \frac{1}{k}\binom{2k}{k} \equiv -\frac{8}{3}\,H(1) \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \frac{4}{5}\left(\frac{H(1)}{p}+2p\,H(3)\right) \pmod{p^4}.$$

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2. Old and new results concerning multiple harmonic sums

We define the *multiple harmonic sum* as

$$H(a_1,a_2,\ldots,a_r;n)=\sum_{1\leqslant k_1<k_2<\cdots<k_r\leqslant n} \frac{1}{k_1^{a_1}k_2^{a_2}\cdots k_r^{a_r}},$$

where $n\geqslant r>0$ and $(a_1,a_2,\ldots,a_r)\in(\mathbb{N}^*)^r$. The values of many harmonic sums modulo a power of prime p are well known, and they are usually expressed in terms of Bernoulli numbers B_n. These are the results which we need later, where we simply write $H(a_1,a_2,\ldots,a_r)$ when the upper summation bound n equals $p-1$:

(i) (see [9]) for any prime $p>5$ we have

$$H(1)\equiv -\frac{1}{2}p\,H(2)\equiv p^2\left(2\frac{B_{p-3}}{p-3}-\frac{B_{2p-4}}{2p-4}\right) \pmod{p^4},$$

$$H(3)\equiv -\frac{3}{2}p\,H(4)\equiv 6p^2\,\frac{B_{p-5}}{p-5} \pmod{p^3},$$

$$H(5)\equiv 0 \pmod{p^2} \quad \text{and} \quad H\left(2;\frac{p-1}{2}\right)\equiv -7\,\frac{H(1)}{p} \pmod{p^3};$$

(ii) (see [5,14]) for $a,b>0$, and for any prime $p>a+b+1$, we have

$$H(a,b)\equiv \frac{(-1)^b}{a+b}\binom{a+b}{a}B_{p-a-b} \quad \text{and} \quad H(1,1,2)\equiv 0 \pmod{p}.$$

The following is a generalization of Wolstenholme's theorem, which improves the congruence modulo p^4 given in Remark 5.1 of [9].

Theorem 2.1. *For any prime $p>5$*

$$H(1)\equiv -\frac{1}{2}p\,H(2)-\frac{1}{6}p^2\,H(3)\equiv p^2\left(\frac{B_{3p-5}}{3p-5}-3\frac{B_{2p-4}}{2p-4}+3\frac{B_{p-3}}{p-3}\right)+p^4\,\frac{B_{p-5}}{p-5} \pmod{p^5}.$$

Proof. Let $m=\varphi(p^5)=p^4(p-1)$. For $r=1,2,3$, Euler's theorem and Faulhaber's formula imply

$$H(r)\equiv\sum_{k=1}^{p-1}k^{m-r}=\frac{B_{m-r+1}(p)-B_{m-r+1}}{m-r+1}\pmod{p^5}.$$

It follows that

$$\sum_{r=1}^{3}\alpha_r p^{r-1}H(r)\equiv\sum_{r=1}^{3}\alpha_r\sum_{k=r}^{m}\frac{p^k B_{m-k}}{k-r+1}\binom{m-r}{k-r}\pmod{p^5}.$$

Because m is even, $B_{m-k}=0$ when $m-k>1$ and k is odd. Moreover, pB_{m-k} is p-integral, and hence the sum modulo p^5 simplifies to

$$\sum_{r=1}^{3}\alpha_r p^{r-1}H(r)\equiv\frac{p^2}{2}(2\alpha_2-\alpha_1)B_{m-2}+\frac{p^4}{4}(-6\alpha_3+4\alpha_2-\alpha_1)B_{m-4}\pmod{p^5}.$$

The right-hand side vanishes if we let $\alpha_1=1$, $\alpha_2=1/2$, and $\alpha_3=1/6$. Finally we use the formulas for $H(2)$ and $H(3)$ modulo p^4 given in Remark 5.1 of [9]. $\square$

The next lemma allows us to expand two kinds of binomial coefficients as combinations of multiple harmonic sums.

Lemma 2.2. *Let n be a positive integer. For $k=1,\dots,n-1$ we have*

$$\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}=(-1)^{k-1}\frac{n}{k}\sum_{j=0}^{k-1}(-n)^jH\big(\{1\}^j;k-1\big),$$

$$\binom{n+k-1}{k}=\frac{n}{k}\binom{n+k-1}{k-1}=\frac{n}{k}\sum_{j=0}^{k-1}n^jH\big(\{1\}^j;k-1\big).$$

Proof. It suffices to use the definition of binomial coefficient:

$$\binom{n-1}{k-1}=\frac{(n-1)\cdots(n-(k-1))}{(k-1)!}=(-1)^{k-1}\prod_{j=1}^{k-1}\left(1-\frac{n}{j}\right)=(-1)^{k-1}\sum_{j=0}^{k-1}(-n)^jH\big(\{1\}^j;k-1\big),$$

and

$$\binom{n+k-1}{k-1}=\frac{(n+k-1)\cdots(n+1)}{(k-1)!}=\prod_{j=1}^{k-1}\left(1+\frac{n}{j}\right)=\sum_{j=0}^{k-1}n^jH\big(\{1\}^j;k-1\big).\qquad\square$$

We already know from (ii) that $H(1,2)\equiv-H(2,1)\equiv B_{p-3}\pmod{p}$. Moreover, because $H(1)H(2)=H(1,2)+H(2,1)+H(3)$ we have $H(1,2)\equiv-H(2,1)$ mod p^2. An identity due to Hernández [4] allows us to disentangle $H(1,2)$ and $H(2,1)$ and prove the following congruence modulo p^2.

Theorem 2.3. *For any prime $p > 3$ we have*

$$H(1,2) \equiv -H(2,1) \equiv -3\,\frac{H(1)}{p^2} \pmod{p^2}.$$

Proof. The following identity, which is valid for $n \geqslant 1$, appears in [4]:

$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{(-1)^{j-1}}{ij} \binom{n}{j}.$$

Letting $n = p$ and using Lemma 2.2, (i), and (ii), we obtain

$$\begin{aligned} H(2) &= p \sum_{1 \leqslant i \leqslant j \leqslant p-1} \frac{(-1)^{j-1}}{ij^2} \binom{p-1}{j-1} + \frac{H(1)}{p} \equiv p \sum_{1 \leqslant i \leqslant j \leqslant p-1} \frac{1 - pH(1; j-1)}{ij^2} + \frac{H(1)}{p} \\ &\equiv pH(1,2) + pH(3) - p^2 \sum_{1 \leqslant i < j \leqslant p-1} \frac{H(1; j-1)}{ij^2} - p^2 H(1,3) + \frac{H(1)}{p} \\ &\equiv pH(1,2) + pH(3) - 2p^2 H(1,1,2) - p^2 H(2,2) - p^2 H(1,3) + \frac{H(1)}{p} \pmod{p^3}. \qquad \square \end{aligned}$$

Finally, we obtain the following extension of a result contained in [13].

Theorem 2.4. *For any prime $p > 5$ we have*

$$\frac{1}{2}\binom{2p}{p} \equiv 1 + 2pH(1) + \frac{2}{3}p^3 H(3) \pmod{p^6}.$$

Proof. Because $H(\{1\}^k; n)$ and $H(k; n)$ are, respectively, the k-th elementary and power-sum symmetric functions in the variables $1, \frac{1}{2}, \dots, \frac{1}{n}$ (see Ex. 7 of Ch. I, §2 in [7]) we find

$$\begin{aligned} 2H\big(\{1\}^2; n\big) &= H^2(1; n) - H(2; n), \\ 6H\big(\{1\}^3; n\big) &= H^3(1; n) - 3H(1; n)H(2; n) + 2H(3; n), \\ 24H\big(\{1\}^4; n\big) &= H^4(1; n) - 6H^2(1; n)H(2; n) + 8H(1; n)H(3; n) + 3H^2(2; n) - 6H(4; n). \end{aligned}$$

Hence, using (i) and Theorem 2.1 we obtain

$$\begin{aligned} H\big(\{1\}^2\big) &\equiv -\frac{1}{2}H(2) \equiv \frac{H(1)}{p} + \frac{1}{6}pH(3) \pmod{p^4}, \\ H\big(\{1\}^3\big) &\equiv \frac{1}{3}H(3) \pmod{p^3}, \\ H\big(\{1\}^4\big) &\equiv -\frac{1}{4}H(4) \equiv \frac{1}{6}\,\frac{H(3)}{p} \pmod{p^2}. \end{aligned}$$

Finally, Lemma 2.2 implies

$$\begin{aligned}\frac{1}{2}\binom{2p}{p} &\equiv 1-2pH(1)+4p^2H\bigl(\{1\}^2\bigr)-8p^3H\bigl(\{1\}^3\bigr)+16p^4H\bigl(\{1\}^4\bigr)\\ &\equiv 1-2pH(1)+4pH(1)+\frac{2}{3}p^3H(3)-\frac{8}{3}p^3H(3)+\frac{8}{3}p^3H(3)\\ &\equiv 1+2pH(1)+\frac{2}{3}p^3H(3) \pmod{p^6},\end{aligned}$$

which is the desired conclusion. □

3. Some preliminary results

We consider the Lucas sequences $\{u_n(x)\}_{n\geqslant 0}$ and $\{v_n(x)\}_{n\geqslant 0}$ defined by the recurrence relations

$$\begin{aligned} &u_0(x)=0, \qquad u_1(x)=1, \quad \text{and} \quad u_{n+1}=x\,u_n(x)-u_{n-1}(x) \quad \text{for } n>0,\\ &v_0(x)=2, \qquad v_1(x)=x, \quad \text{and} \quad v_{n+1}=x\,v_n(x)-v_{n-1}(x) \quad \text{for } n>0.\end{aligned}$$

The corresponding generating functions are

$$U_x(z)=\frac{z}{z^2-xz+1} \quad \text{and} \quad V_x(z)=\frac{2-xz}{z^2-xz+1}.$$

Now we consider two types of sums depending on an integral parameter m. Note that the factor p before the sum is needed because it cancels out the other factor p at the denominator of $\binom{2k}{k}^{-1}$ for $k=(p+1)/2,\dots,p-1$.

Theorem 3.1. *If $m\in\mathbb{Z}$ then for any prime $p\neq 2$,*

$$p\sum_{k=1}^{p-1}\frac{m^k}{k}\binom{2k}{k}^{-1}\equiv\frac{m\,u_p(2-m)-m^p}{2} \pmod{p^2},$$

and

$$p\sum_{k=1}^{p-1}\frac{m^k}{k^2}\binom{2k}{k}^{-1}\equiv\frac{2-v_p(2-m)-m^p}{2p} \pmod{p^2}.$$

Proof. Let $f(z)=1/(1+mz)$. Then

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}\binom{n-1+k}{k-1}\binom{2k}{k}^{-1}(-m)^k &=\frac{1}{2}\sum_{k=1}^{n}\binom{n+k-1}{2k-1}(-m)^k=\frac{1}{2}\bigl[z^n\bigr]f\left(\frac{z}{(1-z)^2}\right)\\ &=-\frac{m}{2}\bigl[z^n\bigr]U_{2-m}(z).\end{aligned}$$

By taking $n = p$, we have

$$\sum_{k=1}^{p-1} \binom{p}{k} \binom{p-1+k}{k-1} \binom{2k}{k}^{-1} (-m)^k = -\frac{mu_p(2-m) - m^p}{2}.$$

On the other hand, by Lemma 2.2 the left-hand side is congruent modulo p^2 to

$$-p \sum_{k=1}^{p-1} \frac{m^k}{k} \big(1 - pH(1; k-1)\big)\big(1 + pH(1; k-1)\big) \binom{2k}{k}^{-1} \equiv -p \sum_{k=1}^{p-1} \frac{m^k}{k} \binom{2k}{k}^{-1} \pmod{p^2}.$$

As for the second congruence, consider

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \binom{n-1+k}{k} \binom{2k}{k}^{-1} (-m)^k &= \sum_{k=0}^{n} \left(\binom{n+k}{2k} - \frac{1}{2} \binom{n+k-1}{2k-1} \right) (-m)^k \\
&= \big[z^n\big] \frac{1}{1-z} f\left(\frac{z}{(1-z)^2} \right) - \frac{1}{2} f\left(\frac{z}{(1-z)^2} \right) \\
&= \frac{1}{2} \big[z^n\big] V_{2-m}(z).
\end{aligned}$$

Again, by taking $n = p$, we have

$$\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} \binom{p-1+k}{k} \binom{2k}{k}^{-1} (-m)^k = -\frac{2 - v_p(2-m) - m^p}{2p}.$$

As before, according to Lemma 2.2 the left-hand side is congruent modulo p^2 to

$$-p \sum_{k=1}^{p-1} \frac{m^k}{k^2} \big(1 - pH(1; k-1)\big)\big(1 + pH(1; k-1)\big) \binom{2k}{k}^{-1} \equiv -p \sum_{k=1}^{p-1} \frac{m^k}{k^2} \binom{2k}{k}^{-1} \pmod{p^2}. \qquad \square$$

For some values of the parameter $x = 2 - m$, Lucas sequences have specific names. Here is a short list of examples which follow directly from the previous theorem.

Corollary 3.2. *For any prime $p \neq 2$, the following congruences hold modulo p^2:*

$$p \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} \equiv \frac{\delta_{p,3}}{2}, \qquad p \sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k}^{-1} \equiv \frac{(\frac{p}{3}) - 1}{2},$$

$$p \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k}^{-1} \equiv \frac{1 - L_p^2}{2p}, \qquad p \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k}^{-1} \equiv \frac{1 - L_p F_p}{2},$$

$$p \sum_{k=1}^{p-1} \frac{2^k}{k^2} \binom{2k}{k}^{-1} \equiv -q_p(2), \qquad p \sum_{k=1}^{p-1} \frac{2^k}{k} \binom{2k}{k}^{-1} \equiv \left(\frac{-1}{p} \right) - 1 - pq_p(2).$$

Here $\delta_{n,k} = 1$ if $n = k$ and it is 0 otherwise, F_n is the n-th Fibonacci number, L_n is the n-th Lucas number, $(\frac{\cdot}{p})$ the Legendre symbol, and $q_p(a) = (a^{p-1} - 1)/p$ the Fermat quotient.

Proof. For $m=1$ we have $u_p(2-m)=(\frac{p}{3})$ and $v_p(2-m)=1-3\delta_{p,3}$. Next, for $m=-1$ we have $u_p(2-m)=F_{2p}=L_pF_p$ and $v_p(2-m)=L_{2p}=L_p^2+2$. Concerning this, note that $L_p\equiv 1$ and $F_p\equiv(\frac{p}{5})$ mod p. Finally, for $m=2$ we have $u_p(2-m)=(\frac{-1}{p})$ and $v_p(2-m)=0$. □

The interested reader may compare some of the previous formulas with the corresponding values of the infinite series (when they converge). For example (see [6]) note the presence of the *golden ratio* in

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\binom{2k}{k}^{-1}=-2\log^2\left(\frac{1+\sqrt{5}}{2}\right)\quad\text{and}\quad\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\binom{2k}{k}^{-1}=-\frac{2}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right),$$

while the analogous congruences involve Fibonacci and Lucas numbers.

By differentiating the generating functions employed in the proof of Theorem 3.1 one can obtain more congruences concerning the expression

$$p\sum_{k=1}^{p-1}\frac{Q(k)\,m^k}{k^2}\binom{2k}{k}^{-1},$$

where $Q(k)$ is a polynomial. For example, it is not hard to show that, for any prime $p>3$,

$$p\sum_{k=1}^{p-1}\frac{1}{C_k}\equiv\frac{2}{3}\left(\frac{p}{3}\right)\pmod{p^2},$$

where $C_k=\binom{2k}{k}/(k+1)$ is the k-th Catalan number.

Increasing the exponent of k at the denominator appears to be a much harder problem, whose natural solution would involve trying to integrate the corresponding generating functions. In the next section we will produce a remarkable example where k appears with exponent 3.

The congruences established in Theorem 3.1 bear a similarity to certain congruences obtained in our joint-work with Zhi-Wei Sun [10] where, contrary as here, the central binomial coefficients appear at the numerator. The following theorem gives a first explanation of this connection, which certainly ought to be further investigated in future studies.

Theorem 3.3. *If $m, r\in\mathbb{Z}$ then for any prime $p\neq 2$ such that p does not divide m we have*

$$p\sum_{k=1}^{p-1}\frac{m^k}{k^r}\binom{2k}{k}^{-1}\equiv\frac{m\,(-1)^{r-1}}{2}\sum_{k=1}^{p-1}\frac{1}{m^k k^{r-1}}\binom{2k}{k}\pmod{p}.$$

Proof. We first show that

$$\frac{p}{k}\binom{2k}{k}^{-1}\equiv\frac{1}{2}\binom{2(p-k)}{p-k}\pmod{p},$$

for $k=1,\dots,p-1$. This is trivial for $k=1,\dots,(p-1)/2$, because both sides have a factor p at the numerator, and so they both vanish modulo p.

Assume now that $k=(p+1)/2,\dots,p-1$. On one hand we find

$$\frac{p}{k}\binom{2k}{k}^{-1}=\frac{(k-1)!}{(p+(2k-p))\cdots(p+1)(p-1)\cdots(p-(p-k-1))}$$
$$\equiv\frac{(k-1)!(-1)^k}{(2k-p)!(p-k-1)!}=(-1)^k\binom{k-1}{p-k-1}\pmod p.$$

On the other hand we have

$$\frac{1}{2}\binom{2(p-k)}{p-k}=\binom{2p-2k-1}{p-k-1}=\frac{(p-(2k+1-p))\cdots(p-(k-1))}{(p-k-1)!}$$
$$\equiv\frac{(-1)^k(k-1)\cdots(2k+1-p)}{(p-k-1)!}=(-1)^k\binom{k-1}{p-k-1}\pmod p.$$

By summing over k and applying Euler's theorem we get

$$p\sum_{k=1}^{p-1}\frac{m^k}{k^r}\binom{2k}{k}^{-1}\equiv\frac{1}{2}\sum_{k=1}^{p-1}\frac{m^k}{k^{r-1}}\binom{2(p-k)}{p-k}=\frac{1}{2}\sum_{k=1}^{p-1}\frac{m^{p-k}}{(p-k)^{r-1}}\binom{2k}{k}$$
$$\equiv\frac{m\,(-1)^{r-1}}{2}\sum_{k=1}^{p-1}\frac{1}{m^k k^{r-1}}\binom{2k}{k}\pmod p.\qquad\square$$

4. Proofs of the main results

We are finally ready to prove the congruences announced in the introduction.

Theorem 4.1. *For any prime $p>5$ we have*

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\binom{2k}{k}^{-1}\equiv\frac{1}{3}\frac{H(1)}{p}\pmod{p^3},$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\binom{2k}{k}^{-1}\equiv-\frac{2}{5}\frac{H(1)}{p^2}\pmod{p^3}.$$

Proof. According to Lemma 2.2 we have

$$p\binom{p-1+k}{k}^{-1}\binom{p-1}{k}^{-1}\equiv(-1)^k k\left(\sum_{j=0}^{3}p^jH\big(\{1\}^j;k-1\big)\right)^{-1}\left(\sum_{j=0}^{3}(-p)^jH\big(\{1\}^j;k\big)\right)^{-1}$$
$$\equiv(-1)^k k\left(1+\frac{p}{k}+p^2H(2;k)+\frac{p^3}{k}H(2;k)\right)\pmod{p^4},$$

where for the second congruence we have used the relations

$$H\big(\{1\}^j;k\big)=H\big(\{1\}^j;k-1\big)+\frac{1}{k}H\big(\{1\}^{j-1};k-1\big)\quad\text{for } j\geqslant 1,$$

and

$$H(1;k)^2=2H(1,1;k)+H(2;k).$$

The rest of the proof depends heavily on two curious identities which play an important role in Apèry's work (see [2] and [12]). The first one is:

$$\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}^{-1}=-\frac{2}{3}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}-\frac{(-1)^n}{3}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{n+k}{k}^{-1}\binom{n}{k}^{-1}.$$

Letting $n=p-1$, from (i) we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}&=-H(2)+\frac{1}{2}H\left(2;\frac{p-1}{2}\right)\equiv 2\frac{H(1)}{p}+\frac{1}{2}\left(-7\frac{H(1)}{p}\right)\\&\equiv-\frac{3}{2}\,\frac{H(1)}{p}\pmod{p^3}.\end{aligned}$$

From (i), (ii), and Theorem 2.3 we get

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{p-1+k}{k}^{-1}\binom{p-1}{k}^{-1}&\equiv\frac{1}{p}\sum_{k=1}^{p-1}\left(\frac{1}{k}+\frac{p}{k^2}+\frac{p^2}{k}H(2;k)+\frac{p^3}{k^2}H(2;k)\right)\\&\equiv\frac{H(1)}{p}+H(2)+p\big(H(2,1)+H(3)\big)+p^2\big(H(2,2)+H(4)\big)\\&\equiv(1-2+3)\frac{H(1)}{p}\equiv 2\frac{H(1)}{p}\pmod{p^3}.\end{aligned}$$

Hence

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\binom{2k}{k}^{-1}\equiv-\frac{2}{3}\left(-\frac{3}{2}\,\frac{H(1)}{p}\right)-\frac{1}{3}\left(2\,\frac{H(1)}{p}\right)\equiv\frac{1}{3}\,\frac{H(1)}{p}\pmod{p^3}.$$

The second identity is:

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\binom{2k}{k}^{-1}=-\frac{2}{5}\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{5}\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\binom{n+k}{k}^{-1}\binom{n}{k}^{-1}.$$

Let $n=p-1$. Since $2H(2,2)=H(2)^2-H(4)\equiv -H(4) \bmod p^2$, then from (i) and (ii) we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\binom{p-1+k}{k}^{-1}\binom{p-1}{k}^{-1}&\equiv\frac{1}{p}\sum_{k=1}^{p-1}\left(\frac{1}{k^2}+\frac{p}{k^3}+\frac{p^2}{k^2}H(2;k)+\frac{p^3}{k^3}H(2;k)\right)\\&\equiv\frac{H(2)}{p}+H(3)+p\big(H(2,2)+H(4)\big)+p^2\big(H(2,3)+H(5)\big)\end{aligned}$$

$$\equiv \frac{H(2)}{p} + H(3) + \frac{p}{2}\left(\frac{4p}{5}B_{p-5}\right) + p^2(-2B_{p-5}+0)$$
$$\equiv \frac{H(2)}{p} + \frac{7}{3}H(3) \pmod{p^3}.$$

Hence by Theorem 2.1

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\binom{2k}{k}^{-1} \equiv -\frac{2}{5}H(3) + \frac{1}{5}\left(\frac{H(2)}{p} + \frac{7}{3}H(3)\right)$$
$$\equiv -\frac{2}{5}\left(-\frac{1}{2}\frac{H(2)}{p} - \frac{1}{6}H(3)\right) \equiv -\frac{2}{5}\frac{H(1)}{p^2} \pmod{p^3}. \qquad \square$$

The duality established in Theorem 3.3 suggests some analogous results for sums involving the central binomial coefficients (not inverted). The former of the following two congruences modulo p^3 appears in [10].

Theorem 4.2. *For any prime $p > 3$*

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k} = -\frac{8}{3}H(1) \pmod{p^4},$$

and for any prime $p > 5$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k} = \frac{4}{5}\left(\frac{H(1)}{p} + 2p\,H(3)\right) \pmod{p^4}.$$

Proof. As regards the first congruence, in [8] Staver proved that for any integer $n \geqslant 1$

$$\sum_{k=1}^{n}\frac{1}{k}\binom{2k}{k} = \binom{2n}{n}\frac{2n+1}{3n^2}\sum_{k=0}^{n-1}\binom{n-1}{k}^{-2}.$$

Now let $n = p$. Taking $a = -2$ in Theorem 1.1 of [11] we obtain

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k} = \frac{1}{p}\binom{2p}{p}\left(\frac{2p+1}{3p}\sum_{k=0}^{p-1}\binom{p-1}{k}^{-2} - 1\right)$$
$$\equiv \frac{2}{p}\binom{2p-1}{p-1}\left(\left(1 - \frac{4}{3}p\,H(1)\right) - 1\right) \equiv -\frac{8}{3}H(1) \pmod{p^4}.$$

In the last step we used the fact that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$, according to Wolstenholme's theorem.

To prove the second congruence we need the following identity, which was conjectured in [3] and then proved in [1], for $n \geqslant 1$:

$$\frac{5}{2}\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4} = \frac{1}{n^2}.$$

If $n = p$, then for $1 \leqslant k < p$ we have

$$\prod_{j=1}^{k-1} \frac{p^4 - j^4}{4p^4 + j^4} = (-1)^{k-1} \prod_{j=1}^{k-1} \frac{1-(p/j)^4}{1+4(p/j)^4} \equiv (-1)^{k-1} \prod_{j=1}^{k-1} \left(1 - \frac{p^4}{j^4}\right)\left(1 - 4\frac{p^4}{j^4}\right)$$
$$\equiv (-1)^{k-1} \prod_{j=1}^{k-1} \left(1 - 5\frac{p^4}{j^4}\right) \equiv (-1)^{k-1}\bigl(1 - 5p^4 H(4; k-1)\bigr) \pmod{p^8}.$$

Using (i) and Theorem 2.4 we find

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \frac{2}{5p^2}\left(\frac{1}{2}\binom{2p}{p}\bigl(1 - 5p^4 H(4)\bigr) - 1\right)$$
$$\equiv \frac{2}{5p^2}\left(\left(1 + 2pH(1) + \frac{2}{3}p^3 H(3)\right)\left(1 + \frac{10}{3}p^4 H(3)\right) - 1\right)$$
$$\equiv \frac{4}{5}\left(\frac{H(1)}{p} + 2p\,H(3)\right) \pmod{p^4},$$

which concludes our proof. $\square$

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